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Weighted Best Local L^p Approximation

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For a weight w with some conditions near the origin the limit of the error

$$\varepsilon^{-n-1}\{f(\varepsilon t) - P_{w,\varepsilon}f(\varepsilon t)\},$$

where $P_{w,\varepsilon}f$ is the weighted best $L^p_{w,\varepsilon}$ approximation of the function f in a class $t^p_{m,w}$, analogous to those considered by Calderón and Zygmund, is characterized. The limit is taken in a norm depending on ε and, with additional assumptions on the weight, in a fixed norm.

1. INTRODUCTION

The interest in the study of local best approximation by algebraic polynomials has recently revived, see, for instance, [2, 7]. Previous papers related to similar problems are [3, 4]. In [4] the authors study the limit behavior of the error $f - P_\varepsilon$, where f is a real analytic function in a neighborhood of the origin and P_ε is the best Tchebycheff approximation with weight w in an ε-neighborhood of zero. For w analytic, $w(0) \neq 0$ and $f^{(m+1)}(0) \neq 0$ (where m is the degree of the approximating polynomials), they proved that the error, if suitably normalized, tends uniformly to the Tchebycheff polynomials of degree $m + 1$. The proof given there requires a clever lemma (see Lemma 1.6, [4]) that cannot be adapted to other L^p norms or to higher dimensions.

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In this paper we show some results on the convergence of the error and its L^p norm. Those results are contained in Theorems 1 and 2. We require a smoothness condition on the function f similar to those considered by Calderón and Zygmund in [1]. We assume a prescribed behavior of the average of the weight function w near the origin, see (2.1). This allows us to consider as a weight function, for instance, negative and fractionary powers. In Theorem 1 we study the convergence using norms depending on ε, whereas in Theorem 2 we consider a fixed norm instead. We point out, see Remark 4, that our method also gives the results obtained in [4].

2. Notation and Results

We shall consider functions defined on $\mathbb{R}^n$. We say that a function w is a weight if it is non-negative and locally integrable. Given a positive real number ε, we write

$$W(\varepsilon) := \int_{|t| \leqslant \varepsilon} w(t)\, dt;$$

we always assume $W(\varepsilon) > 0$. Given a function f and $1 < p < \infty$ we denote

$$\|f\|_{(p,w,\varepsilon)} := \left(W(\varepsilon)^{-1} \int_{|t| \leqslant \varepsilon} |f(t)|^p\, w(t)\, dt\right)^{1/p}.$$

We say that $f \in L^p_{w,\varepsilon}$ if $\|f\|_{(p,w,\varepsilon)} < \infty$. Throughout this paper we consider as approximating class the space π^m of algebraic polynomials with real coefficients of degree less than or equal to m. It is well known that given $f \in L^p_{w,\varepsilon}$ there exists an unique $P_{w,\varepsilon} f \in \pi^m$ such that

$$\|f - P_{w,\varepsilon} f\|_{(p,w,\varepsilon)} = \inf_{P \in \pi^m} \|f - P\|_{(p,w,\varepsilon)}.$$

The polynomial $P_{w,\varepsilon} f$ is called the best approximation of f.

We shall be interested in the asymptotic behavior, for ε tending to zero, of the following errors

$$E_\varepsilon f := \varepsilon^{-m-1}(f(\varepsilon t) - P_{w,\varepsilon} f(\varepsilon t))$$

$$N_\varepsilon f := \|E_\varepsilon f\|_{(p, w_\varepsilon, 1)},$$

where $w_\varepsilon(t) = \varepsilon^n W(\varepsilon)^{-1}\, w(\varepsilon t)$. Note that $W_\varepsilon(1) = 1$. The natural space for the study of these errors seems to be $t^p_{m,w}$, i.e., the class of functions $f \in L^p_{w,1}$ such that there exists $T_m \in \pi^m$ satisfying

$$\|f - T_m\|_{(p,w,\varepsilon)} = o(\varepsilon^m).$$

These classes are similar to those introduced in [1].

We shall consider radial weight functions w satisfying the condition

There exist two real numbers A and β such that $A > 0$, $\beta + n > 0$ and

$$W(\varepsilon) = A\varepsilon^{\beta+n}[1 + o(1)], \qquad \text{for } \varepsilon \text{ tending to zero.} \tag{2.1}$$

It is clear that given w there exists at most one pair of numbers (A, β) such that condition (2.1) holds. For a weight w satisfying these hypotheses we shall prove the uniqueness of the polynomial T_m associated to a function $f \in t^p_{m,w}$. Typical examples of such weights are given by $w(t) = |t|^\alpha$, with $\alpha + n > 0$, finite linear combinations of weights of this type and some infinite linear combinations. Given a weight w satisfying (2.1) we denote

$$\tilde{w}(t) := \omega_n^{-1}(\beta + n)\,|t|^\beta,$$

where ω_n stands for the surface area of the unit ball in $\mathbb{R}^n$.

It follows at once that $t^p_{m,w} \subset t^q_{l,w}$ if $l \leqslant m$ and $p \geqslant q$. Thus, if $f \in t^p_{m+1,w}$ then the function $\Phi_{m+1} := T_{m+1} - T_m$ is well defined and homogeneous of degree $m + 1$.

Our main result is contained in the next theorem.

THEOREM 1. *Let w be a weight satisfying* (2.1) *and* $1 < p < \infty$. *Then, if* $f \in t^p_{m+1,w}$ *it follows that*

$$\operatorname*{Lim}_{\varepsilon \to 0} N_\varepsilon f = \|\Phi_{m+1} - P_{\tilde{w},1}\Phi_{m+1}\|_{(p,\tilde{w},1)}, \tag{2.2}$$

$$\operatorname*{Lim}_{\varepsilon \to 0} \|E_\varepsilon f - (\Phi_{m+1} - P_{\tilde{w},1}\Phi_{m+1})\|_{(p,w_\varepsilon,1)} = 0. \tag{2.3}$$

For a radial weight such that for $A > 0$ and $\beta > -n$

$$w(x) = A\,|x|^\beta\,(1 + o(1)) \tag{2.4}$$

holds, it is immediate that it satisfies condition (2.1). In this case, $w_\varepsilon(t) = |t|^\beta\,(1 + o(1))$. Therefore, the norm appearing in (2.3) can be replaced by a fixed weight $|t|^\beta$. On the other hand it is easy to see the existence of a weight for which (2.1) is true but (2.4) does not hold. Thus, it is convenient to find conditions that allow us to replace the norm depending on ε by a norm with a fixed weight. To this end we consider some classes of weights.

We say that a weight w belongs to $a(r)$, $1 < r < \infty$, if there exists a positive constant C such that

$$\left(\varepsilon^{-m}\int_{|t|\leqslant\varepsilon} w(t)\,dt\right)\left(\varepsilon^{-m}\int_{|t|\leqslant\varepsilon} w(t)^{-r'/r}\,dt\right)^{r/r'} \leqslant C, \tag{2.5}$$

for $0 < \varepsilon < 1$ and $r' = r/(r-1)$.

The class $ah(s)$, $1 < s < \infty$, is defined to be the class of the weights w satisfying

$$\left(\varepsilon^{-n} \int_{|t| \leqslant \varepsilon} w(t)^s \, dt\right)^{1/s} \leqslant C\varepsilon^{-n} \int_{|t| \leqslant \varepsilon} w(t) \, dt, \tag{2.6}$$

for $0 < \varepsilon < 1$ and some constant C.

It is easy to check that (2.5) and (2.6) hold for $-n < \beta < n(r-1)$ and $1 < s < \infty$ if the weight w satisfies (2.4). The classes $a(r)$ are a kind of local analog of the classes A_r of Muckenhoupt. It is known that if $w \in A_r$, then there exists s such that w satisfies (2.6) for every ball, see for instance the survey article [5]. The difference between $a(r)$ and A_r is illustrated by the following example. Let $a_n^i := 2^{-n-2}3 + i2^{-2^n}$, where $i = 1, -1$ and n a non-negative integer. Define

$$w(x) := \begin{cases} 4^{-n} & \text{if } \; a_n^{-1} < x < a_n^1 \\ 1 & \text{if } \; a_n^1 \leqslant x \leqslant a_{n-1}^{-1}. \end{cases}$$

It can be readily verified that w satisfies (2.1) but not (2.4). Moreover w belongs to the classes $a(r)$ and $ah(s)$ for $1 < r, s < \infty$, but w does not belong to any A_p, $1 < p < \infty$. In fact, if $w \in A_p$ for some p, there exists $C > 0$ such that

$$\int_{|x-t| \leqslant 2\varepsilon} w(t) \, dt \leqslant C \int_{|x-t| \leqslant \varepsilon} w(t) \, dt,$$

for every $\varepsilon > 0$ and every x (see [5]). In order to see that the above inequality does not hold it suffices to consider $x = 2^{-n} \cdot 3/4$ and $\varepsilon = 2^{-2^n}$.

We can now state the result concerning the replacement of the norm depending on ε by a fixed one.

THEOREM 2. *Let w be a weight satisfying* (2.1) *and assume that for some* $1 < r$, $s < \infty$, $w \in a(r) \cap ah(s)$. *Let u be a weight belonging to* $ah(s)$, *consider* $f \in t^p_{m+1,w}$ *with* $1 < p < \infty$ *and set* $q = p/s'r$. *Then* $E_\varepsilon f$ *converges to* $\Phi_{m+1} - P_{\tilde{w},1}\Phi_{m+1}$ *in the norm* $L^q_{u,1}$ *when* ε *tends to zero, where the error curve and the polynomials of best approximation are obtained with respect to the norm* L^p *and weight* w.

We shall make some comments on the range of validity and extensions of these Theorems.

Remark 1. Theorem 1 remains valid if $p = 1$, if we restrict ourselves to functions of one real variable which are continuous in a neighborhood of zero (for such functions the best approximation polynomial is unique [6]) the proof of this fact is similar to that of the case $1 < p < \infty$. ∎

Remark 2. For $p = \infty$, we consider again continuous functions of one variable. Let

$$\|f\|_{(\infty, w, \varepsilon)} := \max_{|x| \leqslant \varepsilon} |f(x)\, w(x)|\, W(\varepsilon)^{-1},$$

where $W(\varepsilon) = \max_{|x| \leqslant \varepsilon} w(x)$. Let us assume that w satisfies (2.4) for some $\beta \geqslant 0$. Moreover, suppose that for f the following smoothness property holds: there exists $T \in \pi^{m+1}$ such that

$$\max_{|x| \leqslant \varepsilon} |f(x) - T(x)|\, |x|^{\beta} = o(\varepsilon^{m+1+\beta}).$$

Note that in the condition above we can interchange $|x|^{\beta}$ by $w(\varepsilon x)\, W(\varepsilon)^{-1}$. Then, if $Q(t) = t^{m+1}$, we have

$$\operatorname*{Lim}_{\varepsilon \to 0} \left\| E_{\varepsilon} f - \frac{f^{(m+1)}(0)}{(m+1)!} (Q - P_{\tilde{w},1} Q) \right\|_{(\infty, |x|^{\beta}, 1)} = 0.$$

The proof of the fact follows the lines of the proof of Theorem 1, but it is somehow simpler. ∎

Remark 3. Let us suppose f in $C^{m+1}(I)$, where $I = [-1, 1]$. Let w be continuous and satisfying the conditions of Theorem 2. It is known that there exist $m+1$ points $x_i(\varepsilon)$; $x_0(\varepsilon) < x_1(\varepsilon) < \cdots < x_m(\varepsilon)$ in the closed interval $[-\varepsilon, \varepsilon]$, such that $f(x_i(\varepsilon)) = P_{w,\varepsilon} f(x_i(\varepsilon))$. If $Q(t) = t^{m+1}$ there exist $m+1$ points $x_0 < x_1 < \cdots < x_m$, laying in I, uniquely determined by $\tilde{w}$ and the L^p norm, $1 < p < \infty$, such that $Q(x_i) = P_{\tilde{w},1} Q(x_i)$. In particular, we have

$$Q(t) - P_{\tilde{w},1} Q(t) = (t - x_0)(t - x_1) \cdots (t - x_m).$$

With the further assumption that $f^{(m+1)}(0) \neq 0$, it follows

$$x_i(\varepsilon) = \varepsilon(x_i + o(1)), \qquad i = 0, 1, ..., m.$$

In fact, according to Theorem 2,

$$\operatorname*{Lim}_{\varepsilon \to 0} \left\| E_{\varepsilon} f - \frac{f^{(m+1)}(0)}{(m+1)!} (Q - P_{\tilde{w},1} Q) \right\|_{(q, \tilde{w}, 1)} = 0.$$

Using the formula for the remainder of the Lagrange interpolation polynomial, we have

$$f(x) - P_{w,\varepsilon} f(x) = \frac{f^{(m+1)}(\xi_{\varepsilon})}{(m+1)!} (x - x_0(\varepsilon))(x - x_1(\varepsilon)) \cdots (x - x_m(\varepsilon)),$$

where $\xi_\varepsilon \in (-\varepsilon, \varepsilon)$. Or else

$$E_\varepsilon f(t) = \frac{f^{(m+1)}(\xi_\varepsilon)}{(m+1)!}\left(t - \frac{x_0(\varepsilon)}{\varepsilon}\right)\left(t - \frac{x_1(\varepsilon)}{\varepsilon}\right)\cdots\left(t - \frac{x_m(\varepsilon)}{\varepsilon}\right).$$

Therefore,

$$\operatorname*{Lim}_{\varepsilon\to 0}\left(t - \frac{x_0(\varepsilon)}{\varepsilon}\right)\left(t - \frac{x_1(\varepsilon)}{\varepsilon}\right)\cdots\left(t - \frac{x_m(\varepsilon)}{\varepsilon}\right)$$
$$= (t - x_0)(t - x_1)\cdots(t - x_m).$$

Since $|\varepsilon^{-1}x_i(\varepsilon)| \leqslant 1$, there exists a subsequence $\{\varepsilon_k\}$, such that $\varepsilon_k^{-1}x_i(\varepsilon_k)$ tends to $\bar{x}_i$; $\bar{x}_i \leqslant \bar{x}_{i+1}$. From the equality

$$(t - x_0)(t - x_1)\cdots(t - x_m) = (t - \bar{x}_0)(t - \bar{x}_1)\cdots(t - \bar{x}_m),$$

it follows that $x_i = \bar{x}_i$. This implies the statement that $x_i(\varepsilon) = \varepsilon(x_i + o(1))$. This Remark remains true for $p = \infty$, if we assume the hypotheses of Remark 2. ▮

Remark 4. Given a function f and a weight w, both of them continuous in $[-1, 1]$, it is well known that there exists a Tchebycheff alternation, that is a sequence of $m+2$ points $\{x_i(\varepsilon)\}_{i=0}^{m+1}$ contained in $[-\varepsilon, \varepsilon]$, such that

$$[f(x_i(\varepsilon)) - P_{w,\varepsilon} f(x_i(\varepsilon))]\, w(x_i(\varepsilon))\, W(\varepsilon)^{-1}$$

assume the value of $\|f - P_{w,\varepsilon} f\|_{(\infty,w,\varepsilon)}$ in an alternating fashion. We can take $x_i(\varepsilon) < x_{i+1}(\varepsilon)$. Besides, it is clear that there exists at most one Tchebycheff alternation $-1 \leqslant x_0 < x_1 < \cdots < x_{m+1} \leqslant 1$, for $Q - P_{\tilde{w},1}Q$. Following the lines of [4], we consider the error

$$e_\varepsilon f(t) := \frac{[f(\varepsilon t) - P_{w,\varepsilon} f(\varepsilon t)]}{f(x_i(\varepsilon)) - P_{w,\varepsilon} f(x_i(\varepsilon))}\cdot\left(\frac{w(x_i(\varepsilon))}{W(\varepsilon)}\right)^{-1},$$

where i is equal to 0 or 1 and it is chosen in such a way as to make the denominator positive. It turns out that $e_\varepsilon f(t)$ is well defined if f is not a polynomial of degree m restricted to the interval $[-\varepsilon, \varepsilon]$. Then, if $f \in C^{m+1}$ and $f^{(m+1)}(0) \neq 0$, the error $e_\varepsilon f$ tends to $(Q - P_{\tilde{w},1}Q)/\|Q - P_{\tilde{w},1}Q\|_{(\infty,\tilde{w},1)}$ in the norm $L^\infty_{\tilde{w},1}$. In fact, as we pointed out in Remark 2, $E_\varepsilon f$ tends to $f^{(m+1)}(0)(Q - P_{\tilde{w},1}Q)/(m+1)!$ in $L^\infty_{\tilde{w},1}$. On the other hand there exists a subsequence $\{\varepsilon_k\}$ tending to zero, which defines the numbers $\bar{x}_i = \lim_{k\to\infty} x_i(\varepsilon_k)\varepsilon_k^{-1}$. Therefore

$$\pm(Q(\bar{x}_i) - P_{\tilde{w},1}Q(\bar{x}_i))\,|\bar{x}_i|^\beta = \|Q - P_{\tilde{w},1}Q\|_{(\infty,\tilde{w},1)} > 0.$$

From this it follows that $\{\bar{x}_i\}_{i=0}^{m+1}$ is a Tchebycheff alternation for $Q - P_{\tilde{w},1} Q$. From the uniqueness we obtain that the $\operatorname{Lim}_{\varepsilon \to 0} x_i(\varepsilon)\, \varepsilon^{-1}$ exists and is equal to x_i. This proves the claim. ∎

3. Proof of the Results

We begin by observing some simple facts. Let f^ε be defined by $f^\varepsilon(t) := f(\varepsilon t)$, then

$$\|f\|_{(p,w,\varepsilon)} = \|f^\varepsilon\|_{(p,w_\varepsilon,1)}. \tag{3.1}$$

From this and the uniqueness of the polynomial of best approximation, it follows

$$(P_{w,\varepsilon} f)^\varepsilon = P_{w_\varepsilon,1} f^\varepsilon. \tag{3.2}$$

Also, $P_{w,\varepsilon}$ satisfies the homogeneity property:

$$P_{w,\varepsilon} \lambda f = \lambda P_{w,\varepsilon} f, \qquad \lambda \in \mathbb{R}. \tag{3.3}$$

Let E be a real vector space. Assume that there exists a family of norms $\|\cdot\|_\varepsilon$, for $0 \leqslant \varepsilon < 1$, satisfying

$$\operatorname*{Lim}_{\varepsilon \to 0} \|f\|_\varepsilon = \|f\|_0, \qquad \text{for} \quad f \in E; \tag{3.4}$$

there exists a C greater than zero and a fixed norm $\|\cdot\|$ such that

$$C^{-1} \|f\| \leqslant \|f\|_\varepsilon \leqslant C \|f\|,$$

for every f in a finite dimensional subspace M. (3.5)

Under the conditions above, we have:

Let $f \in E$ and let $P_\varepsilon f$ be an element of M such that

$$\|P - P_\varepsilon f\|_\varepsilon = \inf\{\|f - P\|_\varepsilon : P \in M\}.$$

Assume $P_0 f$ is uniquely determined. Then, $P_\varepsilon f$ converges to $P_0 f$ when ε tends to zero. (3.6)

For the proof of Theorem 1, we need the following lemma.

Lemma 1. *Let w be a weight satisfying condition* (2.1). *Let $P \in \pi^m$ and $1 \leqslant p < \infty$. Then*

$$\big|\, \|P\|_{(p,w_\varepsilon,1)} - \|P\|_{(p,\tilde{w},1)} \big| \leqslant o(1)\, \|P\|_{(p,\tilde{w},1)},$$

where $o(1)$ depends on n, m, p and the weight w only.

Proof. If we denote by

$$f(\rho):=\int_{|t|=1}|P(\rho t)|^p\,dt,$$

then

$$I_\varepsilon:=\|P\|^p_{(p,w_\varepsilon,1)}=\varepsilon^n W(\varepsilon)^{-1}\int_{|t|\leqslant 1}|P(t)|^p\,w(\varepsilon t)\,dt$$

$$=W(\varepsilon)^{-1}\int_0^\varepsilon \rho^{n-1}w(\rho)f(\rho/\varepsilon)\,d\rho.$$

An integration by parts gives

$$I_\varepsilon=(\omega_n W(\varepsilon))^{-1}\,W(\rho)f(\rho/\varepsilon)|_0^\varepsilon-(\varepsilon\omega_n W(\varepsilon))^{-1}\int_0^\varepsilon W(\rho)f'(\rho/\varepsilon)\,d\rho$$

$$=\omega_n^{-1}f(1)-(\varepsilon\omega_n W(\varepsilon))^{-1}A\int_0^\varepsilon \rho^{\beta+n}f'(\rho/\varepsilon)\,d\rho$$

$$+(\varepsilon W(\varepsilon))^{-1}\int_0^\varepsilon o(\rho^{\beta+n})f'(\rho/\varepsilon)\,d\rho$$

$$=:J_1+J_2+J_3.$$

Another integration by parts lead us to

$$J_2=-(\varepsilon\omega_n W(\varepsilon))^{-1}\,\varepsilon f(\rho/\varepsilon)\,A\rho^{\beta+n}|_0^\varepsilon$$

$$+(\omega_n W(\varepsilon))^{-1}A(\beta+n)\int_0^\varepsilon f(\rho/\varepsilon)\,\rho^{\beta+n-1}\,d\rho$$

$$=-J_1(1+o(1))+\omega_n^{-1}(\beta+n)\int_0^1 f(\rho)\,\rho^{\beta+n-1}\,d\rho\cdot(1+o(1)).$$

Then

$$J_1+J_2=J_1o(1)+\|P\|^p_{(p,\tilde w,1)}(1+o(1)).$$

On the other hand

$$f'(\rho)=p\int_{|t|=1}|P(\rho t)|^{p-1}\,sg(P(\rho t))(\nabla P)(\rho t)\cdot t\,dt.$$

Using the Markov inequality, we have

$$|f'(\rho)|\leqslant C\max_{|t|\leqslant 1}|P(t)|^p=C\,\|P\|^p,$$

where $0\leqslant\rho<1$ and the constant C does not depend on P.

Since $\|P\| \leqslant C\|P\|_{(p,\tilde{w},1)}$, for some $C = C(n, m, p, \tilde{w})$, we get

$$|J_3| \leqslant o(1)\|P\|^p_{(p,\tilde{w},1)}.$$

Similarly

$$|J_1| \leqslant C\|P\|^p_{(p,\tilde{w},1)}.$$

Collecting estimates we obtain

$$\begin{aligned}|I_\varepsilon - \|P\|^p_{(p,\tilde{w},1)}| &= |o(1) J_1 + o(1)\|P\|^p_{(p,\tilde{w},1)} + J_3| \\ &\leqslant o(1)\|P\|^p_{(p,\tilde{w},1)}. \quad \blacksquare\end{aligned}$$

From Lemma 1 it follows immediately the equivalence among the norms $\|\cdot\|_{(p,w_\varepsilon,1)}$ and $\|\cdot\|_{(p,\tilde{w},1)}$ on the class π^m, independently of ε. For future references we establish

Let $P \in \pi^m$. Then, there exists a constant C independent of P and ε such that

$$C^{-1}\|P\| \leqslant \|P\|_{(p,w_\varepsilon,1)} \leqslant C\|P\|, \tag{3.7}$$

where

$$\|P\| := \max_{|t| \leqslant 1} |P(t)|.$$

This last remark allows us to show that given a function f in $t^p_{m,w}$ the polynomial $T_m \in \pi^m$ such that

$$\|f - T_m\|_{(p,w,\varepsilon)} = o(\varepsilon^m),$$

is unique. In fact, calling P the difference between two such polynomials, then $\|P\|_{(p,w,\varepsilon)} = o(\varepsilon^m)$. But, taking into account (3.1) and (3.7) we have $\|P\| = o(\varepsilon^m)$, which implies $P = 0$.

Proof of Theorem 1. Since $\Phi_{m+1}(x) := T_{m+1}(x) - T_m(x) = \sum_{|\alpha|=m+1} C_\alpha x^\alpha$, is homogeneous of degree $m+1$ and using (3.2) and (3.3) it follows

$$E_\varepsilon \Phi_{m+1} = \Phi_{m+1} - P_{w_\varepsilon,1}\Phi_{m+1}.$$

In order to proof the convergence of this error, we observe that by (3.7) $\|\cdot\|_{(p,w_\varepsilon,1)}$ and $\|\cdot\|$ considered on the class π^{m+1} are equivalent independently of ε. Besides, by Lemma 1, $\|P\|_{(p,w_\varepsilon,1)}$ converges to $\|P\|_{(p,\tilde{w},1)}$ when ε tends to zero. Therefore by using (3.6) with $E = \pi^{m+1}$ and $M = \pi^m$ we have $\|P_{w_\varepsilon,1}\Phi_{m+1} - P_{\tilde{w},1}\Phi_{m+1}\|$ goes to zero for ε tending to zero. Hence

$$\operatorname*{Lim}_{\varepsilon \to 0} \|E_\varepsilon \Phi_{m+1} - (\Phi_{m+1} - P_{\tilde{w},1}\Phi_{m+1})\| = 0. \tag{3.8}$$

Setting $R_{m+1}=f-T_{m+1}$ and since N_ε is sublinear, we get

$$|N_\varepsilon f-N_\varepsilon T_{m+1}|\leqslant N_\varepsilon R_{m+1}.$$

From (3.8), using that $N_\varepsilon T_{m+1}=N_\varepsilon \Phi_{m+1}$ and

$$N_\varepsilon R_{m+1}\leqslant \varepsilon^{-m-1}\,\|f-T_{m+1}\|_{(p,w,\varepsilon)}=o(1),$$

we obtain

$$\operatorname*{Lim}_{\varepsilon\to 0} N_\varepsilon f=\|\Phi_{m+1}-P_{\tilde{w},1}\Phi_{m+1}\|_{(p,\tilde{w},1)},$$

which proves (2.2).

In order to show the second part of the theorem we write

$$E_\varepsilon f=\varepsilon^{-m-1}(f^\varepsilon-T^\varepsilon_m(f))-H_\varepsilon,$$

where $H_\varepsilon:=\varepsilon^{-m-1}P_{w_\varepsilon,1}(f^\varepsilon-T^\varepsilon_m)$. We have the following direct estimate for H_ε

$$\begin{aligned}\|H_\varepsilon\|_{(p,w_\varepsilon,1)}&\leqslant N_\varepsilon f+\varepsilon^{-m-1}\,\|f-T_m\|_{(p,w,\varepsilon)}\\&\leqslant 2\varepsilon^{-m-1}\,\|f-T_{m+1}\|_{(p,w,\varepsilon)}+2\sum_{|\alpha|=m+1}|C_\alpha|.\end{aligned}$$

Since $f\in t^p_{m+1,w}$, we have that $\|H_\varepsilon\|_{(p,w_\varepsilon,1)}$ is bounded in ε. By (3.7) $\|H_\varepsilon\|$ is also bounded in ε. Therefore, there exists a subsequence ε_k, tending to zero, and a polynomial $H\in\pi^m$ such that H_{ε_k} tends to H. We shall show that H_ε converges itself, by proving that the polynomial H does not depend on the particular subsequence chosen. In fact, from the equality

$$E_\varepsilon f-\Phi_{m+1}+H=\varepsilon^{-m-1}R^\varepsilon_{m+1}(f)+H-H_\varepsilon,$$

it follows that

$$\operatorname*{Lim}_{k\to 0}\|E_{\varepsilon_k}f-(\Phi_{m+1}-H)\|_{(p,w_{\varepsilon_k},1)}=0. \tag{3.9}$$

Now, for $P\in\pi^{m+1}$, we have

$$\begin{aligned}\|E_\varepsilon f\|_{(p,w_\varepsilon,1)}&=\varepsilon^{-m-1}\,\|f^\varepsilon-T^\varepsilon_m(f)-P_{w_\varepsilon,1}(f^\varepsilon-T^\varepsilon_m(f))\|_{(p,w_\varepsilon,1)}\\&\leqslant\varepsilon^{-m-1}\,\|f^\varepsilon-T^\varepsilon_m(f)-\varepsilon^{m+1}P\|_{(p,w_\varepsilon,1)}\\&=\|\varepsilon^{-m-1}(f^\varepsilon-T^\varepsilon_{m+1}(f))+\Phi_{m+1}-P\|_{(p,w_\varepsilon,1)}.\end{aligned}$$

From this, (3.9) and Lemma 1 we get

$$\|\Phi_{m+1}-H\|_{(p,\tilde{w},1)}\leqslant\|\Phi_{m+1}-P\|_{(p,\tilde{w},1)}.$$

The inequality above assures us the existence of the limit of H_ε and that it is equal to $P_{\tilde{w},1}\Phi_{m+1}$. Therefore (3.9) it is also true when the limit is taken for ε going to zero. ∎

The proof of theorem 2 is an easy consequence of the following lemma.

LEMMA 2. *Let u and w be weights such that both of them belong to $ah(s)$ for some $1<s<\infty$. Moreover, suppose $w \in a(r)$ for some $1<r<\infty$. If $1<p<\infty$ and $q=p/s'r$, then*

$$\|f\|_{(q,u,\varepsilon)} \leqslant C\,\|f\|_{(p,w,\varepsilon)},$$

for $0<\varepsilon<1$ and a constant C independent of f and ε.

Before going into the details of the proof we note that Lemma 2 allows us to transfer the smoothness property with weight w to an analogous property with weight u. More precisely, with the notation of Lemma 2, if $f \in t^p_{m,w}$ then $f \in t^q_{m,u}$.

Proof. Denote $I := \|f\|^q_{(q,u,\varepsilon)}$, then

$$I \leqslant \|f\|^q_{(q,w,\varepsilon)} + \int_{|t|\leqslant\varepsilon} |f|^q\,|W(\varepsilon)^{-1}\,w(t) - U(\varepsilon)^{-1}\,u(t)|\,dt$$
$$= I_1 + I_2.$$

We estimate I_2, by using Hölder inequality

$$I_2 \leqslant \left[\int_{|t|\leqslant\varepsilon} |f|^{qs'}\,dt\right]^{1/s'} \cdot \left[\int_{|t|\leqslant\varepsilon} |W(\varepsilon)^{-1}\,w(t) - U(\varepsilon)^{-1}\,u(t)|^s\,dt\right]^{1/s}$$
$$=: I_3 \cdot I_4.$$

From the fact that w and u satisfy (2.6), it results that

$$I_4 \leqslant C\varepsilon^{-n/s'}.$$

By (2.5) and Hölder inequality, we get

$$I_3 \leqslant \|f\|^q_{(p,w,\varepsilon)} \cdot \left(\int_{|t|\leqslant\varepsilon} (W(\varepsilon)^{-1}\,w(t))^{-r'/r}\,dt\right)^{1/s'r'}$$
$$\leqslant C\,\|f\|^q_{(p,w,\varepsilon)} \cdot \varepsilon^{n/s'}.$$

Therefore,

$$I_2 \leqslant C\,\|f\|^q_{(p,w,\varepsilon)}.$$

Since $\|f\|_{(q,w,\varepsilon)}$ is less than or equal to $\|f_{(p,w,\varepsilon)}$, the lemma is proved. ∎

Proof of Theorem 2. From the proof of Theorem 1, we can write the equality

$$E_\varepsilon f - \Phi_{m+1} + P_{\tilde{w},1}\Phi_{m+1} = \varepsilon^{-m-1}R^{\varepsilon}_{m+1}(f) + P_{\tilde{w},1}\Phi_{m+1} - H_\varepsilon.$$

Since $\tilde{w} \in ah(s)$ for any s, $1 < s < \infty$, and H_ε converges to $P_{\tilde{w},1}\Phi_{m+1}$, this equality and Lemma 2 yield the Theorem. ∎

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